Group table and Sudoku puzzles

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Sudoku Introduction

Every $n \times n$ Sudoku has three rules:

1. Every row must contain exactly the numbers from 1 to n, without repetitions;
2. Every column must contain exactly the numbers from 1 to n, without repetitions;
3. If $n = k^2$ is a perfect square, then every (non-overlapping) $k \times k$ subgrid must contain exactly the numbers from 1 to n, without repetitions.

Group table

Every group (multiplication) table will automatically satisfy the first two rules. But it will not satisfy the third rule. Here is an example of a 4×4 group table.

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

If we change the symbols for this group from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$, then it looks like a Sudoku

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

Here is an example of a 9×9 group table.

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
3	4	5	6	7	8	9	1	2
4	5	6	7	8	9	1	2	3
5	6	7	8	9	1	2	3	4
6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6
8	9	1	2	3	4	5	6	7
9	1	2	3	4	5	6	7	8

Here is another example of a 9×9 group table.

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5

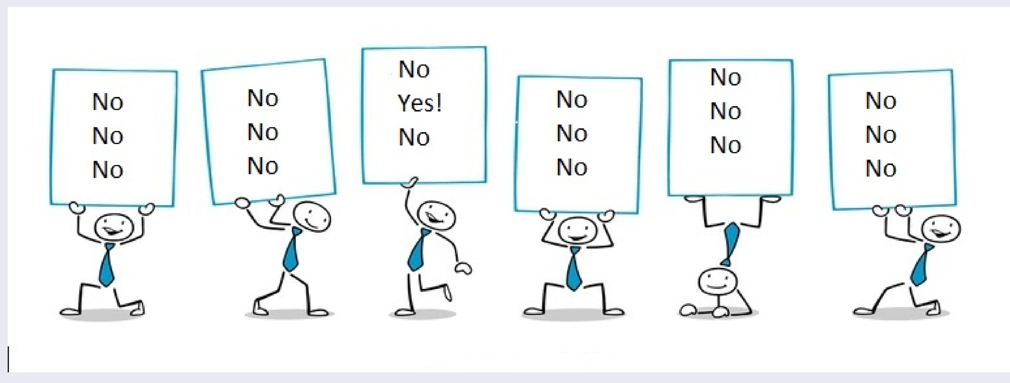
What if we allow row/column switching?

Then it is possible to satisfy all three Sudoku rules. Here is an example of a group table with row/column switching. As a matter of fact, we only switched R2 with R4, R3 with R7, R6 with R8

1	2	3	4	5	6	7	8	9
4	5	6	7	8	9	1	2	3
7	8	9	1	2	3	4	5	6
2	3	1	5	6	4	8	9	7
5	6	4	8	9	7	2	3	1
8	9	7	2	3	1	5	6	4
3	1	2	6	4	5	9	7	8
6	4	5	9	7	8	3	1	2
9	7	8	3	1	2	6	4	5

% of Sudokus are induced by group?

Conclusion: Some Sudokus are induced this way, but extremely rare.
Approximately one out of 21 billion Sudokus is induced by group. Let us use a metaphor to explain this chance.



Algorithmic approach

Extreme rare $\implies$ infeasible to check by human.

With the help of High Performance Computing in WTAMU, we were able to find 29 group-induced ones out of ~ 622 billion Sudokus in 48 hours.

Why not check all of them?

There are about 6.7 trillion billion (10^{21}) Sudokus. It would take the super-computer $\sim 2.1 \times 10^{10}$ seconds, or 680 years.

How does the algorithm work?

1	2	3	**6**	7	8	9	**4**	5
5	8	4	**2**	3	9	7	**6**	1
9	6	7	**1**	4	5	3	**2**	8
3	7	2	**4**	6	1	5	**8**	9
6	9	1	**5**	8	3	2	**7**	4
4	5	8	**7**	9	2	6	**1**	3
8	3	6	**9**	2	4	1	**5**	7
2	1	9	**8**	5	7	4	**3**	6
7	4	5	**3**	1	6	8	**9**	2

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1	2	3	**4**	7	8	9	**6**	5
5	8	4	**6**	3	9	7	**2**	1
9	6	7	**2**	4	5	3	**1**	8
3	7	2	**8**	6	1	5	**4**	9
6	9	1	**7**	8	3	2	**5**	4
4	5	8	**1**	9	2	6	**7**	3
8	3	6	**5**	2	4	1	**9**	7
2	1	9	**3**	5	7	4	**8**	6
7	4	5	**9**	1	6	8	**3**	2

1	2	3	4	5	6	7	8	9
5	8	4	6	1	2	3	9	7
9	6	7	2	8	1	4	5	3
3	7	2	8	9	4	6	1	5
6	9	1	7	4	5	8	3	2
4	5	8	1	3	7	9	2	6
8	3	6	5	7	9	2	4	1
2	1	9	3	6	8	5	7	4
7	4	5	9	2	3	1	6	8

--→

1	2	3	4	5	6	7	8	9
2	1	9	3	6	8	5	7	4
3	7	2	8	9	4	6	1	5
4	5	8	1	3	7	9	2	6
5	8	4	6	1	2	3	9	7
6	9	1	7	4	5	8	3	2
7	4	5	9	2	3	1	6	8
8	3	6	5	7	9	2	4	1
9	6	7	2	8	1	4	5	3

Then we check associativity for this "supposed-to-be" group table.

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